

THE NUMBER OF SUBGROUP CHAINS OF NON-ABELIAN GROUP $Z_p \times S_4$, p IS ANY PRIME

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Manuscript received on the 17/06/23 and accepted on the 25/07/23

ABSTRACT

In this paper, we determined the number of subgroup chains of the Cartesian product of a cyclic group of order p and a symmetric group of degree four using a computational technique induced by isomorphism classes of subgroups of G

Keywords: Lattice of Subgroup; Symmetric Group; Subgroup chains; Fuzzy Subgroup

Mathematics Subject Classification (2020): 20B30, 20B35, 20N25, 20E15.

INTRODUCTION

One of the most important problems in combinatorial group theory is to counting the number of subgroup chains of a group. . Many papers have treated various aspects of this problem in the last few years (Ogiugo et al., 2018, Adebisi et al., 2020, Ardekan and Davvaz, 2017) The problem of counting chains of subgroups of a given group G has received attention from researchers related to classifying fuzzy subgroups of G under a certain type of equivalence relation. The number of chains of subgroups was computed by giving its one variable generating function (see Tarnăuceanu and Bentea, 2008) . J.M. Oh in his paper determined the number of subgroups of a finite cyclic group of $4n$ by giving its multi variables generating function (Oh, 2012). The study of the number of chains of subgroups in the lattice of subgroups for larger groups in classical group theory is very interesting, because it will give rise to potential applications in solving the combinatorial problem of group testing for future pandemics within a larger population.

Group testing methods utilize constrained testing resources more efficiently by pooling specimens together, potentially allowing larger populations to be screened with fewer tests. Similarly, a study of the number of chains of subgroups of G in the subgroup lattice in which only certain subgroups of G can be tested. In this paper, we established the number of subgroup chains of the Cartesian product of a cyclic group of order p and a symmetric group of degree four

PRELIMINARIES

A chain of subgroups of G is a set of subgroups of G totally ordered by set inclusion. The study of chains of subgroups in this paper describes the set of all chains of subgroups of G that end in G .

In this way, suppose that the group G is finite and let $\mu: G \mapsto [0,1]$ be a fuzzy subgroup of G . Then $\mu(G)$ is a finite set, Put $\mu(G) = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ and assume that $\alpha_1 < \alpha_2 < \dots < \alpha_k$. Then μ determines the following chain of subgroups of G which ends in G :

$$\{e\} \subseteq \mu G \alpha_1 \subset \mu G \alpha_2 \dots \subset \mu G \alpha_k = G$$

Moreover, for any $x \in G$ and $i = \overline{1, k}$, we have

$$\mu(x) = \alpha_i \Leftrightarrow \alpha_i = \max\{j \mid x \in \mu G \alpha_j\} \Leftrightarrow x \in \mu G \alpha_i / \mu G \alpha_{1-k}$$

we set $\mu G_{\alpha_0} = 0$, then $\mu(e) = \alpha_i$

An adequate equivalence relation on the set of all fuzzy subgroups of the group must be defined to obtain a meaningful classification. Without any equivalence relation on fuzzy subgroups of group G , the

number of fuzzy subgroups is infinite, even for the trivial group

We say that μ is equivalent to ν , written as $\mu \sim \nu$

$$\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y), \forall x, y \in G,$$

and

$$\mu(x) = 0 \Leftrightarrow \nu(x) = 0, \forall x \in G$$

Note that the condition $\mu(x) = 0$ holds if and only if $\nu(x) = 0$ simply says that the supports of μ and ν are equal and two fuzzy subgroups μ, ν of G is said to be distinct if $\mu \not\sim \nu$

This also shows that there exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of

subgroups of G which end in G . So, the problem of counting all distinct fuzzy subgroups of G can be translated into a combinatorial problem on the subgroup lattice of G that is finding the number of all chains of subgroups of G that terminate in G (Volf, 2004)

Let G be a finite group and $\delta(G)$ be the number of chains of subgroups of G that terminate in G

(1) $G_1 \subset G_2 \subset \dots \subset G_k = G$ with $G_1 \neq \{e\}$

(2) $\{e\} \subset G_2 \subset \dots \subset G_k = G$

The numbers of chains of types (1) and (2) are equal. So

$$\delta(G) = 2x_k$$

This problem of counting all chains of subgroups of G depends entirely on the lattice of subgroups of G and not on the group itself. This leads to a more general problem

Then

$$\delta(G) = \sum_{\text{distinct } H \in \text{Iso}(G)} \delta(H) \times n(H)$$

Let G be any finite group and $\delta(G)$ be the number of subgroup chains of group G that terminates in G .

where (i) $\text{Iso}(G)$ is the set of representatives

of isomorphism classes of subgroups of G

(ii) $n(H)$ denotes the size of the isomorphism class with representative H

(iii) we set $\delta(H_e) = \delta(H_k) = 1$, for which H_e is the trivial group of and H_k is the improper subgroup of G (see, Ogiugo and Amit, 2020)

RESULTS

1. The number of subgroup chains of $Z_5 \times S_4$

Let G be $Z_5 \times S_4$ is a non-abelian group of order 120, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$\{\{e\}, 1\}, [Z_2, 9], [Z_3, 4], [Z_4, 3], [Z_5, 1], [Z_{10}, 9], [Z_{15}, 4], [Z_{20}, 3]$

$[Z_2 \times Z_2, 4], [Z_{10} \times Z_2, 4], [Z_5 \times S_3, 4], [Z_5 \times A_4, 1], [Z_5 \times D_8, 3]$

$[A_4, 1], [D_8, 3], [S_3, 4], [S_4, 1]$ and $[Z_5 \times S_4, 1]$,

then

$$\begin{aligned}\delta(G) = & 1 + 9 * \delta(Z_2) + 4 * \delta(Z_3) + 3 * \delta(Z_4) + \delta(Z_5) + 9 * \delta(Z_{10}) + 4 * \delta(Z_{15}) + 3 \\ & * \delta(Z_{20}) + 4 * \delta(Z_2 \times Z_2) + 4 * \delta(Z_{10} \times Z_2) + 4 * \delta(Z_5 \times S_3) + \delta(Z_5 \times A_4) \\ & + 3 * \delta(Z_5 \times D_8) + \delta(A_4) + 3 * \delta(D_8) + 4 * \delta(S_3) + \delta(S_4) + 1 = 1656\end{aligned}$$

2. The number of subgroup chains of $Z_7 \times S_4$

Let G be $Z_7 \times S_4$, it has the following set of representatives of isomorphism classes of subgroups with their sizes: $\{e\}, 1, [Z_2, 9], [Z_3, 4], [Z_4, 3], [Z_7, 1], [Z_{14}, 9], [Z_{21}, 4], [Z_{28}, 3], [Z_2 \times Z_2, 4], [Z_{14} \times Z_2, 4], [Z_7 \times S_3, 4], [Z_7 \times A_4, 1], [Z_7 \times D_8, 3], [A_4, 1], [D_8, 3], [S_3, 4], [S_4, 1]$ and $[Z_7 \times S_4, 1]$, then

$$\begin{aligned}\delta(G) = & 1 + 9 * \delta(Z_2) + 4 * \delta(Z_3) + 3 * \delta(Z_4) + \delta(Z_7) + 9 * \delta(Z_{14}) + 4 * \delta(Z_{21}) + 3 \\ & * \delta(Z_{28}) + 4 * \delta(Z_2 \times Z_2) + 4 * \delta(Z_{14} \times Z_2) + 4 * \delta(Z_7 \times S_3) + \delta(Z_7 \times A_4) \\ & + 3 * \delta(Z_7 \times D_8) + \delta(A_4) + 3 * \delta(D_8) + 4 * \delta(S_3) + \delta(S_4) + 1 = 1656\end{aligned}$$

3. The number of subgroup chains of $Z_{11} \times S_4$

Let G be $Z_{11} \times S_4$, it has the following set of representatives of isomorphism classes of subgroups with their sizes: $\{e\}, 1, [Z_2, 9], [Z_3, 4], [Z_4, 3], [Z_{11}, 1], [Z_{22}, 9], [Z_{33}, 4], [Z_{44}, 3], [Z_2 \times Z_2, 4], [Z_{22} \times Z_2, 4], [Z_{11} \times S_3, 4], [Z_{11} \times A_4, 1], [Z_{11} \times D_8, 3], [A_4, 1], [D_8, 3], [S_3, 4], [S_4, 1]$ and $[Z_{11} \times S_4, 1]$, then

$$\begin{aligned}\delta(G) = & 1 + 9 * \delta(Z_2) + 4 * \delta(Z_3) + 3 * \delta(Z_4) + \delta(Z_{11}) + 9 * \delta(Z_{22}) + 4 * \delta(Z_{33}) + 3 \\ & * \delta(Z_{44}) + 4 * \delta(Z_2 \times Z_2) + 4 * \delta(Z_{22} \times Z_2) + 4 * \delta(Z_{11} \times S_3) + \delta(Z_{11} \times A_4) \\ & + 3 * \delta(Z_{11} \times D_8) + \delta(A_4) + 3 * \delta(D_8) + 4 * \delta(S_3) + \delta(S_4) + 1 = 1656\end{aligned}$$

4. The number of subgroup chains of $Z_p \times A_4$, $p \geq 5$

Let G be $Z_5 \times A_4$, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$\{e\}, 1, [Z_2, 3], [Z_3, 4], [Z_5, 1], [Z_{10}, 3], [Z_{15}, 4], [Z_2 \times Z_2, 1], [Z_{10} \times Z_2, 1], [A_4, 1]$ and $[Z_5 \times A_4, 1]$, then

$$\begin{aligned}\delta(G) = & 1 + 3 * \delta(Z_2) + 4 * \delta(Z_3) + \delta(Z_5) + 3 * \delta(Z_{10}) + 4 * \delta(Z_{15}) + \delta(Z_2 \times Z_2) \\ & + \delta(Z_{10} \times Z_2) + \delta(A_4) + 1 = 128\end{aligned}$$

Let G be $Z_7 \times A_4$, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$\{e\}, 1, [Z_2, 3], [Z_3, 4], [Z_7, 1], [Z_{14}, 3], [Z_{21}, 4], [Z_2 \times Z_2, 1], [Z_{14} \times Z_2, 1], [A_4, 1]$ and $[Z_7 \times A_4, 1]$, then

$$\begin{aligned}\delta(G) = & 1 + 3 * \delta(Z_2) + 4 * \delta(Z_3) + \delta(Z_7) + 3 * \delta(Z_{14}) + 4 * \delta(Z_{21}) + \delta(Z_2 \times Z_2) \\ & + \delta(Z_{14} \times Z_2) + \delta(A_4) + 1 = 128\end{aligned}$$

Let G be $Z_p \times A_4$, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$\{e\}, 1, [Z_2, 3], [Z_3, 4], [Z_p, 1], [Z_{10}, 3], [Z_{2p}, 4], [Z_2 \times Z_2, 1], [Z_{2p} \times Z_2, 1], [A_4, 1]$ and $[Z_p \times A_4, 1]$, then

$$\begin{aligned}\delta(G) = & 1 + 3 * \delta(Z_2) + 4 * \delta(Z_3) + \delta(Z_p) + 3 * \delta(Z_{2p}) + 4 * \delta(Z_{3p}) + \delta(Z_2 \times Z_2) \\ & + \delta(Z_{2p} \times Z_2) + \delta(A_4) + 1 = 128\end{aligned}$$

Proposition 4.1

Suppose that G be the Cartesian product of group $Z_p \times A_4$, where p is prime number, then $\delta(G) = 128$ if $p \geq 5$.

5. The number of subgroup chains of $Z_p \times D_8$, $p \geq 3$

Let G be $Z_3 \times D_8$, it has the following set of representatives of isomorphism classes of subgroups with their sizes: $\{\{e\}, 1\}$, $[Z_2, 5]$, $[Z_3, 1]$, $[Z_4, 1][Z_6, 5]$, $[Z_{12}, 1]$, $[Z_2 \times Z_2, 2]$, $[Z_6 \times Z_2, 2]$, $[D_8, 1]$ and $[Z_3 \times D_8, 1]$, then

$$\delta(G) = 1 + 5 * \delta(Z_2) + \delta(Z_3) + \delta(Z_4) + 5 * \delta(Z_6) + \delta(Z_{12}) + 2 * \delta(Z_2 \times Z_2) + 2 * \delta(Z_6 \times Z_2) + \delta(D_8) + 1 = 184$$

Let G be $Z_5 \times D_8$, it has the following set of representatives of isomorphism classes of subgroups with their sizes: $\{\{e\}, 1\}$, $[Z_2, 5]$, $[Z_5, 1]$, $[Z_4, 1][Z_{10}, 5]$, $[Z_{20}, 1]$, $[Z_2 \times Z_2, 2]$, $[Z_{10} \times Z_2, 2]$, $[D_8, 1]$ and $[Z_5 \times D_8, 1]$, then

$$\delta(G) = 1 + 5 * \delta(Z_2) + \delta(Z_5) + \delta(Z_4) + 5 * \delta(Z_{10}) + \delta(Z_{20}) + 2 * \delta(Z_2 \times Z_2) + 2 * \delta(Z_{10} \times Z_2) + \delta(D_8) + 1 = 184$$

Let G be $Z_p \times D_8$, it has the following set of representatives of isomorphism classes of subgroups with their sizes: $\{\{e\}, 1\}$, $[Z_2, 5]$, $[Z_p, 1]$, $[Z_4, 1][Z_{2p}, 5]$, $[Z_{4p}, 1]$, $[Z_2 \times Z_2, 2]$, $[Z_{2p} \times Z_2, 2]$, $[D_8, 1]$ and $[Z_p \times D_8, 1]$, then

$$\delta(G) = 1 + 5 * \delta(Z_2) + \delta(Z_p) + \delta(Z_4) + 5 * \delta(Z_{2p}) + \delta(Z_{4p}) + 2 * \delta(Z_2 \times Z_2) + 2 * \delta(Z_{2p} \times Z_2) + \delta(D_8) + 1 = 184$$

Proposition 5.1

Suppose that G be the Cartesian product of group $Z_p \times D_8$, where p is prime number, then $\delta(G) = 184$ if $p \geq 3$.

6. The number of subgroup chains of $Z_p \times S_4$, $p \geq 5$

Theorem 6.1 (Ogiugo,2021)

The number of the subgroup chains of the group $Z_2 \times S_4$ is 3376.

Theorem 6.2 (Ogiugo,2021)

The number of the subgroup chains of the group $Z_3 \times S_4$ is 1792.

Theorem 6.3

The number of subgroup chains of $Z_p \times S_4$, $p \geq 5$ is 1656.

Proof:

Let G be $Z_p \times S_4$, it has the following set of representatives of isomorphism classes of subgroups with their sizes: $\{\{e\}, 1\}$, $[Z_2, 9]$, $[Z_3, 4]$, $[Z_p, 1]$, $[Z_4, 3][Z_2 \times Z_2, 4]$, $[Z_{2p}, 9]$, $[Z_{3p}, 4]$, $[Z_{4p}, 3]$, $[A_4, 1]$, $[Z_p \times A_4, 1]$, $[(Z_p \times D_8), 3]$, $[D_8, 3]$, $[S_3, 4]$, $[S_4, 1]$, $[(Z_p \times S_3), 4]$, and $[(Z_p \times S_4), 1]$,

$$\text{then, } \delta(Z_p \times S_4) = 1 + \delta(H_e) + 9 * \delta(Z_2) + 4 * \delta(Z_3) + \delta(Z_p) + 3 * \delta(Z_4) + 4 * \delta(Z_2 \times Z_2) + 9 * \delta(Z_{2p}) + 4 * \delta(Z_{3p}) + 3 * \delta(Z_{4p}) + \delta(A_4) + \delta(Z_p \times A_4) + 3 * \delta(Z_p \times D_8) + 3 * \delta(D_8) + 4 * \delta(S_3) + \delta(S_4) + 4 * \delta(Z_p \times S_3) = 1656$$

CONCLUSION

In this paper, the first step in classifying the fuzzy subgroups in the Cartesian product of the cyclic group of order p and a symmetric group of degree 4 is made. It also showed that a fuzzy subgroup is simply a chain of subgroups in the lattice of subgroups based on the natural equivalence relation.

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